

Relativistic Viscous Fluid Dynamics and Non-Equilibrium Entropy

Paul Romatschke

Institute for Nuclear Theory, University of Washington,

Box 351550, Seattle, WA, 98195

E-mail: paulrom@phys.washington.edu

ABSTRACT: Fluid dynamics corresponds to the dynamics of a substance in the long wavelength limit. Writing down all terms in a gradient (long wavelength) expansion up to second order for a relativistic system at vanishing charge density, one obtains the most general (causal) equations of motion for a fluid in the presence of shear and bulk viscosity, as well as the structure of the non-equilibrium entropy current. Requiring positivity of the divergence of the non-equilibrium entropy current relates some of its coefficients to those entering the equations of motion. I comment on possible applications of these results for conformal and non-conformal fluids.

Contents

1. Introduction	1
2. Setup	2
2.1 Conformal Fluids	4
2.2 Non-Conformal Fluids	5
3. Viscous Fluid Dynamics: Energy-Momentum Tensor	6
3.1 Dispersion Relations and Kubo Formula	8
4. Non-Equilibrium Entropy	9
4.1 Conformal Fluids	9
4.2 Non-Conformal Fluids	13
5. Second Order Transport Coefficients: Known Results	14
6. Discussion and Conclusions	16
A. Conformal Transformations	18
B. Useful Identities	19
C. $\nabla_\mu S^\mu_{\text{non-eq}}$ for Non-Conformal Fluids	20

1. Introduction

The theory of fluid dynamics has a long history, starting with its inception by L. Euler in 1755 [1] and generalized to viscous fluids by the works of C. Navier [2] and G. Stokes [3] in the 19th century. Besides being necessary to realistically model the behavior of real fluids in many situations, viscosity is known to be essential for the presence of laminar (or smooth) flows since it dampens the turbulent instability generally inherent to ideal (non-viscous) fluids [4]§26. As can be understood from the work by Navier and Stokes, viscous effects correspond to (first order) gradients of the equilibrium properties of the system (temperature, fluid velocity, etc.).

For relativistic systems, additional complications arise: a first order gradient expansion à la Navier and Stokes leads to a set of fluid dynamics equations that allow faster-than-light signal propagation, violating causality [5]. In the late 20th century, motivated by questions in general relativity, Müller [6], Israel and Stewart [7] showed that by including second order gradient terms, the resulting fluid dynamics equations could be made causal. In the last decade – driven by nuclear physics experiments on relativistic heavy-ion collisions [8, 9, 10, 11] – a program to determine the precise form of the equations for a relativistic viscous fluid was initiated [12, 13, 14, 15]. This program has not been finished, and the present work is meant to provide another step towards its completion.

It should be noted that the question of causality is intimately linked to the property of hyperbolicity of the fluid dynamic theory, i.e. the question of whether a well-defined initial value formulation can be given. While for small perturbations around equilibrium hyperbolicity and causality can be shown for various second order theories, it seems that in the case of strong non-linear out of equilibrium situations hyperbolicity has been rigorously investigated only in so-called divergence type theories [16], e.g. Müller’s theory [6]. In particular, the causality properties of the theory by Israel and Stewart far out of equilibrium remain unknown. This work is based on the more recent interpretation of fluid dynamics as an effective theory of the long-wavelength modes of the system, by construction precluding any application to systems that are far from equilibrium. While the effective theory specified below will share with other second theories the property of causality (and hyperbolicity) for small perturbations around equilibrium, a proof of even hyperbolicity for strong perturbations is beyond the scope of this work.

This article is organized as follows: in section 2, all relevant gradient structures up to second order are listed for a fluid at zero charge density, and the concept of a conformal fluid is introduced. In section 3, the most general form of the energy-momentum tensor for non-conformal fluids is presented, which fixes the equations of motion for a relativistic viscous fluid. Section 4 deals with the most general form of the entropy current for a relativistic fluid out of equilibrium (again, at vanishing charge density), and its divergence. In section 5, I tried to collect all current knowledge about the coefficients multiplying second order gradient structures. Section 6 contains a discussion of the results and the conclusions.

2. Setup

Let us consider matter described by a relativistic quantum-field theory at vanishing charge density (zero chemical potential). In equilibrium, the long-wavelength dynamics

of this system can be described by one scalar, one vector, and one tensor:

$$\varepsilon, u^\mu, g^{\mu\nu},$$

which are the fundamental fluid dynamic variables energy density ε , fluid four velocity u^μ and metric tensor $g_{\mu\nu}$. The quantum field theory is supposed to furnish a relation between the pressure P and ε via the equation of state, $P = P(\varepsilon)$, giving rise to the speed of sound $c_s = \sqrt{dP/d\varepsilon}$ in the fluid.

Using these fundamental degrees of freedom, one can write down the energy-momentum tensor of the fluid (see, e.g. [4]§133),

$$T_{\text{eq}}^{\mu\nu} = \varepsilon u^\mu u^\nu + P \Delta^{\mu\nu}, \quad \Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu, \quad (2.1)$$

where the subscript “eq” denotes equilibrium quantities containing no gradients (zeroth order). Here and in the following the metric sign convention $(-, +, +, +)$ is used, and all the calculations will be done in four space-time dimensions (though a generalization to other number of dimensions should be straightforward). Without the presence of a source, the energy-momentum tensor is covariantly conserved, $\nabla_\mu T^{\mu\nu} = 0$, where ∇_μ denotes the geometric covariant derivative. Projection of this equation leads to the well-known equations of motion for an ideal relativistic fluid,

$$\begin{aligned} u_\nu \nabla_\mu T_{\text{eq}}^{\mu\nu} &= -D\varepsilon - (\varepsilon + P)\nabla \cdot u = 0, \\ \Delta_\nu^\alpha \nabla_\mu T_{\text{eq}}^{\mu\nu} &= (\varepsilon + P)Du^\alpha + \nabla_\perp^\alpha P = 0, \end{aligned} \quad (2.2)$$

where the new notations

$$D \equiv u^\mu \nabla_\mu, \quad \nabla_\perp^\mu \equiv \Delta^{\mu\nu} \nabla_\nu$$

for the comoving time and space derivative were introduced. Using the basic thermodynamic relations $\delta\varepsilon = T\delta s$, $\varepsilon + P = sT$ for the equilibrium entropy density s and temperature T , the equations of motion for the ideal fluid become

$$D \ln s = -\nabla \cdot u, \quad Du^\alpha = -c_s^2 \nabla_\perp^\alpha \ln s. \quad (2.3)$$

This implies that not all first order gradients of the fundamental degrees of freedom are independent: time derivatives may (up to higher order gradient corrections) always be recast as space derivatives. Therefore, to first order in gradients, the only independent structures one can write down are $\nabla_\alpha^\perp \ln s$ and $\nabla_\alpha^\perp u_\beta$ (no coordinate-invariant first order gradient of the metric tensor exists). For later convenience, gradients are sorted into three classes: scalars, vectors orthogonal to u^μ , and symmetric traceless tensors orthogonal to u^μ . To first order in gradients, there is one of each class:

$$(\nabla \cdot u), \quad \nabla_\alpha^\perp \ln s, \quad \sigma_{\alpha\beta} \equiv \nabla_\alpha^\perp u_\beta + \nabla_\beta^\perp u_\alpha - \frac{2}{3} \Delta_{\alpha\beta} (\nabla \cdot u).$$

To second order in gradients, the independent structures are $\nabla_\alpha^\perp \nabla_\beta^\perp \ln s$, $\nabla_\alpha^\perp \nabla_\beta^\perp u_\mu$, $\nabla_\alpha^\perp \ln s \nabla_\beta^\perp \ln s$, $\nabla_\alpha^\perp u_\mu \nabla_\beta^\perp u_\nu$, $\nabla_\beta^\perp u_\mu \nabla_\alpha^\perp \ln s$ and the Riemann tensor (cf. [17]§3.4)

$$R_{\mu\sigma\nu}^\lambda \equiv \partial_\sigma \Gamma_{\mu\nu}^\lambda - \partial_\nu \Gamma_{\mu\sigma}^\lambda + \Gamma_{\mu\nu}^\kappa \Gamma_{\kappa\sigma}^\lambda - \Gamma_{\mu\sigma}^\kappa \Gamma_{\kappa\nu}^\lambda,$$

where $\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$. Using the Ricci tensor $R_{\mu\nu} = R_{\mu\lambda\nu}^\lambda$ and Ricci scalar $R = R^\mu_\mu$, one can build 7 independent scalars to second order in gradients,

$$\nabla_\alpha^\perp \nabla_\perp^\alpha \ln s, \quad \nabla_\alpha^\perp \ln s \nabla_\perp^\alpha \ln s, \quad \sigma_{\alpha\mu} \sigma^{\alpha\mu}, \quad \Omega_{\alpha\mu} \Omega^{\alpha\mu}, \quad (\nabla \cdot u)^2, \quad u^\mu u^\nu R_{\mu\nu}, \quad R, \quad (2.4)$$

where $\Omega_{\mu\nu} = \frac{1}{2} (\nabla_\mu^\perp u_\nu - \nabla_\nu^\perp u_\mu)$ is the fluid vorticity which was used in the decomposition

$$\nabla_\mu^\perp u_\nu = \frac{1}{2} \sigma_{\mu\nu} + \Omega_{\mu\nu} + \frac{(\nabla \cdot u)}{3} \Delta_{\mu\nu}.$$

Furthermore, one can build 6 independent vectors orthogonal to the fluid velocity:

$$\nabla_\alpha^\perp \sigma^{\alpha\mu}, \quad \nabla_\alpha^\perp \Omega^{\alpha\mu}, \quad \sigma^{\alpha\mu} \nabla_\alpha^\perp \ln s, \quad (\nabla \cdot u) \nabla_\perp^\mu \ln s, \quad \Delta^{\alpha\mu} u^\beta R_{\alpha\beta}, \quad (2.5)$$

where one should note that $\Delta^{\alpha\mu} u^\beta R_{\alpha\beta}$ contains – and therefore is used instead of – the vector $\nabla_\perp^\mu (\nabla \cdot u)$ (see appendix B).

Finally, there are 8 independent symmetric traceless tensors orthogonal to the fluid velocity. These have been found already in Ref. [14] for the case of conformal fluids (see below for the definition of a conformal fluid). For a general relativistic fluid at vanishing charge density, the 8 independent tensors are given by

$$\begin{aligned} \nabla_\perp^{<\mu} \ln s \nabla_\perp^{\nu>} \ln s, \quad \nabla_\perp^{<\mu} \nabla_\perp^{\nu>} \ln s, \quad \sigma^{\mu\nu} (\nabla \cdot u), \quad \sigma^{<\mu}_\lambda \sigma^{\nu>\lambda}, \\ \sigma^{<\mu}_\lambda \Omega^{\nu>\lambda}, \quad \Omega^{<\mu}_\lambda \Omega^{\nu>\lambda}, \quad u_\alpha u_\beta R^{\alpha<\mu\nu>\beta}, \quad R^{<\mu\nu>}, \end{aligned} \quad (2.6)$$

where for a second rank tensor $A^{\mu\nu}$

$$\langle A^{\mu\nu} \rangle = A^{<\mu\nu>} \equiv \frac{1}{2} \Delta^{\mu\alpha} \Delta^{\nu\beta} (A_{\alpha\beta} + A_{\beta\alpha}) - \frac{1}{3} \Delta^{\mu\nu} \Delta^{\alpha\beta}.$$

2.1 Conformal Fluids

For certain situations it is advantageous to consider the simplified case of a fluid without bulk viscosity. Since the bulk viscosity coefficient ζ is related to the conformal anomaly T^μ_μ via a Kubo relation, a prime example for such a case is a system that exhibits conformal symmetry, or covariance under local Weyl rescalings of the metric:

$$g_{\mu\nu} \rightarrow e^{-2w} g_{\mu\nu}, \quad (2.7)$$

where $w = w(x^\mu)$ is the local scale factor. I will refer to the fluid description of a system that obeys conformal covariance simply as “conformal fluid”. An example for a

conformal quantum field theory is the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM). Note that in curved space the Weyl anomaly [18] would in general break conformal symmetry, since, e.g., $T_\mu^\mu \propto R^2$. However, this breaking occurs only at fourth order in gradients, suggesting that for a fluid dynamic expansion up to third order in gradients the Weyl anomaly can be ignored [14].

Conformal covariance implies that objects transform homogeneously under Weyl rescalings, meaning that gradients of the scale factor w have to cancel. To lowest order, consistency requires [14]

$$s \rightarrow e^{3w} s, \quad u^\mu \rightarrow e^w u^\mu.$$

To first order in gradients, there are no scalars or vectors¹ but one symmetric tensor that transforms homogeneously under Weyl rescalings: $\sigma_{\mu\nu}$ (see appendix A). To second order in gradients, there are three conformal scalars,

$$\begin{aligned} \mathcal{S}_1 &= \sigma_{\mu\nu} \sigma^{\mu\nu}, \quad \mathcal{S}_2 = \Omega_{\mu\nu} \Omega^{\mu\nu}, \\ \mathcal{S}_3 &= c_s^2 \nabla_\mu^\perp \nabla_\perp^\mu \ln s + \frac{c_s^4}{2} \nabla_\mu^\perp \ln s \nabla_\perp^\mu \ln s - \frac{1}{2} u_\alpha u_\beta R^{\alpha\beta} - \frac{1}{4} R + \frac{1}{6} (\nabla \cdot u)^2, \end{aligned} \quad (2.8)$$

two conformal vectors orthogonal to u^μ ,

$$\mathcal{V}_1^\mu = \nabla_\alpha^\perp \sigma^{\alpha\mu} + 2c_s^2 \sigma^{\alpha\mu} \nabla_\alpha^\perp \ln s - \frac{u^\mu}{2} \sigma_{\alpha\beta} \sigma^{\alpha\beta}, \quad \mathcal{V}_2^\mu = \nabla_\alpha^\perp \Omega^{\mu\alpha} + u^\mu \Omega_{\alpha\beta} \Omega^{\alpha\beta}, \quad (2.9)$$

and five conformal symmetric traceless tensors orthogonal to u^μ ,

$$\begin{aligned} \mathcal{O}_1^{\mu\nu} &= R^{<\mu\nu>} - c_s^2 (2 \nabla_\perp^{<\mu} \nabla_\perp^{\nu>} \ln s + \sigma^{\mu\nu} (\nabla \cdot u) - 2c_s^2 \nabla_\perp^{<\mu} \ln s \nabla_\perp^{\nu>} \ln s), \\ \mathcal{O}_2^{\mu\nu} &= R^{<\mu\nu>} - 2u_\alpha u_\beta R^{\alpha<\mu\nu>\beta}, \\ \mathcal{O}_3^{\mu\nu} &= \sigma^{<\mu}_\lambda \sigma^{\nu>\lambda}, \quad \mathcal{O}_4^{\mu\nu} = \sigma^{<\mu}_\lambda \Omega^{\nu>\lambda}, \quad \mathcal{O}_5^{\mu\nu} = \Omega^{<\mu}_\lambda \Omega^{\nu>\lambda}. \end{aligned} \quad (2.10)$$

These will be the building blocks of the energy-momentum tensor and entropy current for conformal fluids.

2.2 Non-Conformal Fluids

For more general fluids that do not obey conformal invariance, all possible gradients can contribute. In particular, to first order in gradients there are one scalar and vector,

$$(\nabla \cdot u), \quad \nabla^\mu \ln s, \quad (2.11)$$

in addition to the tensor already found for the conformal case.

¹For non-vanishing charge density, parity-breaking effects may occur in the fluid. In this case, also terms such as $\epsilon^{\mu\nu\alpha\beta} u_\nu \Omega_{\alpha\beta}$ are allowed.

At second order, there are four additional scalars,

$$\mathcal{S}_4 = (\nabla \cdot u)^2, \quad \mathcal{S}_5 = R, \quad \mathcal{S}_6 = \nabla_\mu^\perp \ln s \nabla_\perp^\mu \ln s, \quad \mathcal{S}_7 = u^\alpha u^\beta R_{\alpha\beta}, \quad (2.12)$$

four additional vectors,

$$\mathcal{V}_3^\mu = \Delta^{\mu\beta} u^\alpha R_{\alpha\beta}, \quad \mathcal{V}_4^\mu = \sigma^{\mu\alpha} \nabla_\alpha^\perp \ln s, \quad \mathcal{V}_5^\mu = \Omega^{\mu\alpha} \nabla_\alpha^\perp \ln s, \quad \mathcal{V}_6^\mu = (\nabla \cdot u) \nabla_\perp^\mu \ln s, \quad (2.13)$$

and three additional tensors,

$$\mathcal{O}_6^{\mu\nu} = u_\alpha u_\beta R^{\alpha<\mu\nu>\beta}, \quad \mathcal{O}_7^{\mu\nu} = \frac{(\nabla \cdot u)}{3} \sigma^{\mu\nu}, \quad \mathcal{O}_8^{\mu\nu} = \nabla_\perp^{<\mu} \ln s \nabla_\perp^{\nu>} \ln s. \quad (2.14)$$

3. Viscous Fluid Dynamics: Energy-Momentum Tensor

The energy-momentum tensor for a relativistic viscous fluid can be written as

$$T_{\text{non-eq}}^{\mu\nu} = T_{\text{eq}}^{\mu\nu} + \Pi^{\mu\nu},$$

where the viscous stress tensor $\Pi^{\mu\nu}$ contains correction terms to the ideal energy-momentum tensor due to shear and bulk viscosity. At this point, I recall that for vanishing charge density² the only useful way to define the fluid velocity u^μ is the Landau-Lifshitz condition

$$u_\mu T^{\mu\nu} = \varepsilon u^\nu,$$

which implies $u_\mu \Pi^{\mu\nu} = 0$. The viscous stress tensor is customarily separated into a traceless part ($\pi^{\mu\nu}$) and a part with non-vanishing trace (Π),

$$\Pi^{\mu\nu} = \pi^{\mu\nu} + \Delta^{\mu\nu} \Pi.$$

For conformal fluids, where $T_\mu^\mu = 0$, the trace part vanishes identically and the structure of the traceless part $\pi^{\mu\nu}$ in the fluid dynamic (small gradient) regime is generated by $\sigma^{\mu\nu}$ and the tensors found in Eq. (2.10) (cf. Ref. [14]). For general fluids, also the tensors in Eq. (2.14) contribute. Using instead of $\mathcal{O}_1^{\mu\nu}$ the expression³

$$\langle D\sigma^{\mu\nu} \rangle + \frac{2-3c_s^2}{3} \sigma^{\mu\nu} (\nabla \cdot u) \simeq \mathcal{O}_1^{\mu\nu} - \mathcal{O}_2^{\mu\nu} - \frac{1}{2} \mathcal{O}_3^{\mu\nu} + 2\mathcal{O}_5^{\mu\nu} - 2 \frac{dc_s^2}{d \ln s} \nabla_\perp^{<\mu} \ln s \nabla_\perp^{\nu>} \ln s, \quad (3.1)$$

²For non-vanishing charge density ρ , the presence of the charge current j^μ offers other choices to define u^μ , such as $u_\mu j^\mu = \rho$. In the Landau-Lifshitz frame, since u^μ is defined via the rest-frame of the energy-density, heat diffusion does not exist; but, since in this frame generally $u_\mu j^\mu \neq \rho$, there can be charge diffusion. In the so-called Eckart frame $u_\mu j^\mu = \rho$, and there is heat diffusion instead of charge diffusion, indicating that these concepts are different manifestations of the same phenomenon.

³Note the wrong sign of the \mathcal{O}_5 term in Ref. [14] that has been corrected here.

which is accurate to second order in gradients and amounts to a particular resummation of higher order terms, one finds

$$\begin{aligned}\pi^{\mu\nu} = & -\eta\sigma^{\mu\nu} + \eta\tau_\pi \left[\langle D\sigma^{\mu\nu} \rangle + \frac{\nabla \cdot u}{3} \sigma^{\mu\nu} \right] + \kappa [R^{<\mu\nu>} - 2u_\alpha u_\beta R^{\alpha<\mu\nu>\beta}] \\ & + \lambda_1 \sigma^{<\mu}_{\lambda} \sigma^{\nu>\lambda} + \lambda_2 \sigma^{<\mu}_{\lambda} \Omega^{\nu>\lambda} + \lambda_3 \Omega^{<\mu}_{\lambda} \Omega^{\nu>\lambda} \\ & + \kappa^* 2u_\alpha u_\beta R^{\alpha<\mu\nu>\beta} + \eta\tau_\pi^* \frac{\nabla \cdot u}{3} \sigma^{\mu\nu} + \lambda_4 \nabla^{<\mu} \ln s \nabla^{\nu>} \ln s .\end{aligned}\quad (3.2)$$

The coefficient of the first order gradient term is the familiar shear viscosity coefficient η . The coefficients $\tau_\pi, \tau_\pi^*, \kappa, \kappa^*, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ are “second order” transport coefficients, three of which ($\tau_\pi^*, \kappa^*, \lambda_4$) must be identically zero for conformal fluids since they multiply structures that do not transform homogeneously under Weyl rescalings.

The expression for the trace part Π in the fluid dynamic regime contains all scalars up to second order in gradients, which are given by Eq. (2.8) and Eq. (2.12). Using instead of \mathcal{S}_3 the expression

$$D(\nabla \cdot u) \simeq -\frac{1}{4}\mathcal{S}_1 + \mathcal{S}_2 - \mathcal{S}_3 - \frac{1}{6}\mathcal{S}_4 - \frac{1}{4}\mathcal{S}_5 + \left(\frac{3c_s^4}{2} - \frac{d c_s^2}{d \ln s} \right) \mathcal{S}_6 - \frac{3}{2}\mathcal{S}_7 ,$$

one finds

$$\begin{aligned}\Pi = & -\zeta(\nabla \cdot u) + \zeta\tau_\Pi D(\nabla \cdot u) + \xi_1 \sigma^{\mu\nu} \sigma_{\mu\nu} + \xi_2 (\nabla \cdot u)^2 \\ & + \xi_3 \Omega^{\mu\nu} \Omega_{\mu\nu} + \xi_4 \nabla_\mu^\perp \ln s \nabla_\perp^\mu \ln s + \xi_5 R + \xi_6 u^\alpha u^\beta R_{\alpha\beta} .\end{aligned}\quad (3.3)$$

Here ζ is the familiar bulk viscosity coefficient and $\tau_\Pi, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6$ are second order transport coefficients for non-conformal fluids. As will be shown below, at least two of these second order coefficients (e.g., ξ_5, ξ_6) are completely specified in terms of other transport coefficients. The equations (3.2), (3.3) give the most general structure for the energy-momentum tensor of a relativistic viscous fluid at zero charge density up to second order in gradients.

Similar to the case for ideal fluids, the equations of motion for a viscous fluid are given by $\nabla_\mu T^{\mu\nu} = 0$. In particular, using again the basic thermodynamic relations one finds for $u_\nu \nabla_\mu T^{\mu\nu} = 0$ the result

$$Ds + s(\nabla \cdot u) = -\frac{1}{2T} \pi^{\mu\nu} \sigma_{\mu\nu} - \frac{1}{T} \Pi(\nabla \cdot u) , \quad (3.4)$$

which will be useful in the following.

3.1 Dispersion Relations and Kubo Formula

Considering small perturbations $\delta\varepsilon, \delta u^\mu$ around an equilibrium configuration in flat space with $\varepsilon = \text{const.}$ and $u^\mu = (1, \mathbf{0})$, one obtains dispersion relations for the collective modes along (L) and perpendicular (T) to the perturbation. The analysis is straightforward (see for example Ref. [19] §II.C) and in the fluid dynamic regime ($\omega, k \ll 1$) leads to

$$\begin{aligned}\omega_L(k) &= \pm k c_s - i k^2 \Gamma \mp \frac{k^3}{2 c_s} \left[\Gamma^2 - 2 c_s^2 \left(\frac{2}{3} \frac{\eta \tau_\pi}{\varepsilon + P} + \frac{1}{2} \frac{\zeta \tau_\Pi}{\varepsilon + P} \right) \right] + \mathcal{O}(k^4), \\ \omega_T(k) &= -\frac{i \eta k^2}{\varepsilon + P} + \mathcal{O}(k^4),\end{aligned}\tag{3.5}$$

where it is recalled that transverse and longitudinal perturbations correspond to the shear and sound mode, respectively, and the “sound attenuation length” is given by

$$\Gamma = \left(\frac{2}{3} \frac{\eta}{\varepsilon + P} + \frac{1}{2} \frac{\zeta}{\varepsilon + P} \right).$$

The equations of motion for a viscous fluid are found to be causal if the maximal propagation speeds at high wavenumber k for the sound and shear mode are smaller than the speed of light (cf. [19] §II). Using the dispersion relations, one finds for the propagation speeds in natural units ($\hbar = c = k_B = 1$):

$$\lim_{k \rightarrow \infty} \frac{d \omega_T(k)}{d k} = \sqrt{\frac{\eta}{\tau_\pi(\varepsilon + P)}}, \quad \lim_{k \rightarrow \infty} \frac{d \omega_L(k)}{d k} = \sqrt{c_s^2 + \frac{4}{3} \frac{\eta}{\tau_\pi(\varepsilon + P)} + \frac{\zeta}{\tau_\Pi(\varepsilon + P)}}.\tag{3.6}$$

The values are fixed by the first and second order transport coefficients, or the properties of system in the fluid dynamic (long wavelength) limit, while causality concerns the property of the system for short wavelength perturbations. There is a priori no reason why $\lim_{k \rightarrow \infty} \frac{d \omega_{T,L}}{d k}$ should be less than unity⁴.

The second order transport coefficients are related to thermal correlators via general Kubo formulas. As an example, let us calculate the retarded correlator $G_R^{xy,xy}$ for the energy momentum tensor component T^{xy} . Considering a metric perturbation $\delta g_{\mu\nu}$ with

⁴One could still use the second order equations of motion as a phenomenological model of a relativistic viscous fluid if the above conditions were violated for a particular quantum field theory: adjusting the second order transport coefficients τ_π, τ_Π “by hand” to repair causality, the resulting model would still treat the first order gradients correctly. In a way, something similar is done for numerical simulations of ideal fluids: to dampen the turbulent instability inherent to ideal fluid dynamics, one has to introduce “numerical viscosity” (first order gradient correction terms) to obtain stable evolution for the ideal (zeroth order gradient) fluid. Here second order gradients are needed to obtain stable and causal evolution of a viscous (first order gradient) fluid.

only the non-vanishing component $\delta g_{xy}(t, z)$, the fluid stays at rest and in equilibrium: $\epsilon = \text{const}$, $u^\mu = (1, \mathbf{0})$, since this corresponds to a tensor perturbation. The correlator may then be found from the linear response of T^{xy} to this perturbation (cf. [14]):

$$G_R^{xy,xy} = P - i\eta\omega + \left(\eta\tau_\pi - \frac{\kappa}{2} + \kappa^*\right)\omega^2 - \frac{\kappa}{2}k^2 + \mathcal{O}(\omega^3, k^3).$$

This implies that once τ_π is known, both κ and κ^* can be found by calculating this correlator for a quantum field theory in the appropriate limit.

4. Non-Equilibrium Entropy

For a system in equilibrium the entropy current is simply given by the product of entropy density and fluid velocity, $S_{\text{eq}}^\mu = su^\mu$. For systems out of equilibrium it may be that the entropy current gets modified⁵. It is known from kinetic theory that at zero chemical potential (in the absence of heat/charge diffusion), the first correction to the equilibrium entropy current must be of second order in gradients [20]. In the past, the form of the non-equilibrium entropy current has often been postulated [7]. More recently, a more fundamental approach has been advocated [21, 22] that calls for all structures in a gradient expansion to be allowed. I will follow this approach here, recovering and extending some of the results in Ref. [22].

4.1 Conformal Fluids

For conformal fluids, the entropy current must be built out of elements that are invariant under conformal transformations, which are the three scalars (2.8) and two vectors (2.9):

$$\begin{aligned} S_{\text{non-eq}}^\mu = su^\mu + \frac{A_1}{4}\mathcal{S}_1 u^\mu + A_2\mathcal{S}_2 u^\mu + A_3\left(4\mathcal{S}_3 - \frac{1}{2}\mathcal{S}_1 + 2\mathcal{S}_2\right)u^\mu \\ + B_1\left(\frac{1}{2}\mathcal{V}_1^\mu + \frac{u^\mu}{4}\mathcal{S}_1\right) + B_2(\mathcal{V}_2^\mu - u^\mu\mathcal{S}_2), \end{aligned} \quad (4.1)$$

where the five coefficients $A_{1,2,3}$ and $B_{1,2}$ are (mass dimension one) functions of entropy only and the combinations and prefactors have been chosen such as to facilitate comparison to Ref. [22]. For conformal fluids in three dimensions one has $c_s^2 = \frac{1}{3}$ and $\Pi = 0$, and the absence of a second dimensionful scale leads to the relation $A_i, B_i \sim s^{1/3}$. According to Boltzmann's H-theorem, entropy is never allowed to decrease, so the divergence of the non-equilibrium entropy current should obey

$$\nabla_\mu S_{\text{non-eq}}^\mu \geq 0.$$

⁵I am not aware of any experimental verification of this hypothesis.

The divergence of the entropy current is a physical observable, and as such should transform homogeneously under Weyl rescalings. Explicitly, one can convince oneself that this is the case by writing

$$\nabla_\mu S^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} S^\mu) \rightarrow \frac{e^{4w}}{\sqrt{-g}} \partial_\mu (e^{-4w} \sqrt{-g} e^{4w} S^\mu) .$$

Taking the covariant derivative of Eq. (4.1), the result for the equilibrium part $\nabla_\mu (su^\mu)$ can be read off from Eq. (3.4). A somewhat more lengthy calculation (see appendix B for some useful identities) gives

$$\begin{aligned} \nabla_\mu S_{\text{non-eq}}^\mu &= \frac{1}{2} \nabla_\mu^\perp \nabla_\nu^\perp \sigma^{\mu\nu} (-2A_3 + B_1) + \frac{1}{3} \nabla_\mu^\perp \sigma^{\mu\nu} \nabla_\nu^\perp \ln s (-2A_3 + B_1) \\ &+ \sigma_{\mu\nu} \left[\frac{\eta}{2T} \sigma^{\mu\nu} + R^{\mu\nu} \left(-\frac{\kappa}{2T} + A_3 \right) + u_\alpha u_\beta R^{\alpha<\mu\nu>\beta} \left(\frac{\kappa - \eta\tau_\pi}{T} + A_1 + B_1 - 2A_3 \right) \right. \\ &- \frac{1}{4} \sigma^\mu{}_\lambda \sigma^{\nu\lambda} \left(\frac{2\lambda_1 - \eta\tau_\pi}{T} + A_1 + B_1 - 2A_3 \right) + \frac{1}{3} \nabla_\perp^{<\mu} \nabla_\perp^{\nu>} \ln s \left(\frac{\eta\tau_\pi}{T} - A_1 - 2A_3 \right) \\ &+ \Omega^\mu{}_\alpha \Omega^{\nu\alpha} \left(-\frac{\lambda_3 + 2\eta\tau_\pi}{2T} + A_1 - 2A_2 - 2A_3 + B_1 \right) \\ &\left. + \sigma^{\mu\nu} \frac{(\nabla \cdot u)}{12} \left(\frac{2\eta\tau_\pi}{T} - 2A_1 + 6A_3 - 5B_1 \right) + \frac{1}{9} \nabla_\perp^{<\mu} \ln s \nabla_\perp^{\nu>} \ln s \left(-\frac{\eta\tau_\pi}{T} + A_1 + B_1 \right) \right] , \end{aligned} \quad (4.2)$$

where the conformal fluid property $\tau_\pi^* = \kappa^* = \lambda_4 = 0$ was used. Note that B_2 completely drops out of the divergence of the entropy current because the covariant derivative of the relevant term is vanishing. Positivity of $\nabla_\mu S_{\text{non-eq}}^\mu$ is usually guaranteed by the term $\sigma_{\mu\nu} \sigma^{\mu\nu}$, which is of second order in gradients and hence generally much larger than all the other (third order gradient) terms. However, there is the possibility that $\sigma_{\mu\nu}$ itself is accidentally small, e.g., when considering a fluid velocity field that has very little shear motion in it. In this case it can happen that third order gradients dominate the entropy production, and hence their coefficients must be such that entropy never decreases. This immediately implies

$$B_1 = 2A_3 , \quad (4.3)$$

since the first two terms of Eq. (4.2) could otherwise lead to negative entropy production (this relation was already pointed out in Ref. [22]). Now by the same logic, one would also expect the coefficients of the term $\sigma_{\mu\nu} R^{\mu\nu}$ to vanish, leading to

$$A_3 = \frac{\kappa}{2T} . \quad (4.4)$$

However, there is a possible loophole in the argument leading to Eq.(4.4): it could be that the term $\sigma_{\mu\nu} R^{\mu\nu}$ combines with other terms of second and fourth order in gradients to form a full square, e.g.,

$$\begin{aligned} \sigma_{\mu\nu} \left[\frac{\eta}{2T} \sigma^{\mu\nu} + R^{\mu\nu} \left(A_3 - \frac{\kappa}{2T} \right) \right] &= -\frac{T}{2\eta} \left(A_3 - \frac{\kappa}{2T} \right)^2 R_{\mu\nu} R^{\mu\nu} \\ &+ \frac{\eta}{2T} \left[\sigma_{\mu\nu} + \frac{T}{\eta} \left(A_3 - \frac{\kappa}{2T} \right) R_{\mu\nu} \right] \left[\sigma^{\mu\nu} + \frac{T}{\eta} \left(A_3 - \frac{\kappa}{2T} \right) R^{\mu\nu} \right]. \end{aligned} \quad (4.5)$$

In this scenario (pointed out in Ref. [22]), the coefficient A_3 could be arbitrary and positivity of the entropy current would still be guaranteed if the offending first term on the r.h.s of Eq. (4.5) is offset by another term of fourth order in gradients. There are three possible sources of fourth order gradient terms in Eq. (4.2): third order gradients in $\pi^{\mu\nu}$ times $\sigma_{\mu\nu}$ stemming from $\nabla_\mu (su^\mu)$, first order gradients in $\pi^{\mu\nu}$ times third order gradients stemming from viscous corrections to Eq. (2.3), and finally derivatives of third order gradients in the non-equilibrium entropy current $S_{\text{non-eq}}^\mu$. First note that the term $R_{\mu\nu} R^{\mu\nu}$ cannot be offset by a fourth order gradient term from $\nabla_\mu (su^\mu)$ because — as Eq. (3.4) shows — the latter always involves $\sigma_{\mu\nu}$ which is not expressible in terms of $R_{\mu\nu}$. Secondly, viscous corrections to Eq. (2.3) up to second order in gradients are found to be

$$\begin{aligned} D \ln s &\simeq -(\nabla \cdot u) + \eta \frac{\sigma^{\mu\nu} \sigma_{\mu\nu}}{2sT}, \\ Du^\alpha &\simeq -c_s^2 \nabla_\perp^\alpha \ln s + \frac{\nabla_\mu^\perp (\eta \sigma^{\mu\alpha})}{sT} - c_s^2 \nabla_\mu^\perp \ln s \frac{\eta \sigma^{\mu\alpha}}{sT} - u^\alpha \eta \frac{\sigma^{\mu\nu} \sigma_{\mu\nu}}{2sT}, \end{aligned} \quad (4.6)$$

which shows that —again— the term $R_{\mu\nu} R^{\mu\nu}$ cannot be offset by these contributions. Finally, all third order gradient terms contributing to the entropy current may be found by the same principle as in section 2. Explicitly, one finds as scalars:

- $\sigma_{\mu\nu}$ times all symmetric second order tensors (e.g., $\sigma_{\mu\nu} \mathcal{O}_1^{\mu\nu}$),
- $\nabla_\mu^\perp \ln s$ times all second order vectors (e.g., $\nabla_\mu^\perp \ln s \mathcal{V}_1^\mu$),
- $(\nabla \cdot u)$ times all second order scalars (e.g., $(\nabla \cdot u) \mathcal{S}_1$),
- ∇_\perp^μ acting on all second order vectors (e.g., $\nabla_\mu^\perp \mathcal{V}_1^\mu$),

and as vectors:

- $\nabla_\mu^\perp \ln s$ times all second order tensors (e.g., $\nabla_\alpha^\perp \ln s \mathcal{O}_1^{\mu\alpha}$),
- $(\nabla \cdot u), \sigma_{\mu\nu}, \Omega_{\mu\nu}$ times all second order vectors (e.g., $(\nabla \cdot u) \mathcal{V}_1^\mu, \sigma_\alpha^\mu \mathcal{V}_1^\alpha$),

- $\nabla_\mu^\perp \ln s$ times all second order scalars (e.g., $\nabla_\perp^\mu \ln s \mathcal{S}_1$),
- ∇_\perp^μ acting on all second order scalars (e.g., $\nabla_\perp^\mu \mathcal{S}_1$),
- ∇_\perp^μ acting on all second order tensors (e.g., $\nabla_\alpha^\perp \mathcal{O}_1^{\alpha\mu}$).

One can readily convince oneself that derivatives of third order vectors in the entropy current will not be able to generate a term such as $R_{\mu\nu}R^{\mu\nu}$. For scalars, the only potentially dangerous term is the time derivative of $\sigma_{\mu\nu} \mathcal{O}_2^{\mu\nu}$, because $\mathcal{O}_2^{\mu\nu}$ contains $R_{\mu\nu}$ and $D\sigma_{\mu\nu}$ could potentially contain another $R_{\mu\nu}$. But the explicit expression (3.1) for $D\sigma_{\mu\nu}$ shows that this is not the case, hence no $R_{\mu\nu}R^{\mu\nu}$ term in $\nabla_\mu S_{\text{non-eq}}^\mu$ is generated from any of the possible sources.

As a consequence, the scenario (4.5) cannot be possibly realized for every A_3 . Only the condition (4.4) ensures positivity of the non-equilibrium entropy.

A similar (but slightly more complicated) argument can be made for the terms $u_\alpha u_\beta R^{\alpha<\mu\nu>\beta} u^\gamma u^\delta R_{\gamma<\mu\nu>\delta}$ and $\Omega^{<\mu\alpha} \Omega^{\nu>\alpha} \Omega_{<\mu\beta} \Omega_{\nu>\beta}$: for these terms to be canceled or made positive definite, one has to add the two third-order gradient terms $x_1 \sigma_{\mu\nu} (\mathcal{O}_1^{\mu\nu} - \mathcal{O}_2^{\mu\nu})$ and $x_2 \sigma_{\mu\nu} 2\mathcal{O}_5^{\mu\nu}$ to $S_{\text{non-eq}}^\mu$ with the requirements

$$x_1 \geq \frac{T}{8\eta} \left(\frac{\kappa - \eta\tau_\pi}{T} + A_1 \right)^2, \quad x_2 \geq \frac{T}{8\eta} \left(-\frac{\lambda_3 + 2\eta\tau_\pi}{2T} + A_1 - 2A_2 \right)^2.$$

To cancel the cross-term $u_\alpha u_\beta R^{\alpha<\mu\nu>\beta} \Omega^{<\mu\gamma} \Omega_{\nu>\gamma}$, however, implies the condition

$$x_1 + x_2 = \frac{T}{4\eta} \left(\frac{\kappa - \eta\tau_\pi}{T} + A_1 \right) \left(-\frac{\lambda_3 + 2\eta\tau_\pi}{2T} + A_1 - 2A_2 \right).$$

The resulting inequality relation can only be fulfilled if

$$A_2 = -\frac{2\kappa + \lambda_3}{4T}. \quad (4.7)$$

Unfortunately, I did not find an argument that would fix the value of A_1 in terms of second order transport coefficients. Nevertheless, it seems that a bound on the value of A_1 could be found by extending $\pi^{\mu\nu}$ to third order in gradients, since $x_1 \sigma_{\mu\nu} D(\mathcal{O}_1^{\mu\nu} - \mathcal{O}_2^{\mu\nu})$ involves $D(u_\alpha u_\beta R^{\alpha<\mu\nu>\beta})$ which seemingly has to be canceled exactly by a corresponding expression in $\pi^{\mu\nu}$, fixing x_1 .

To conclude, requiring positivity of the divergence of the non-equilibrium entropy current fixes three of the five possible coefficients (A_2, A_3, B_1). The divergence then takes the form

$$\begin{aligned} \nabla_\mu S_{\text{non-eq}}^\mu &= \frac{\eta}{2T} \sigma_{\mu\nu} \sigma^{\mu\nu} + \frac{\kappa - 2\lambda_1}{4T} \sigma_{\mu\nu} \sigma^\mu{}_\lambda \sigma^{\nu\lambda} \\ &+ \left(\frac{A_1}{2} + \frac{\kappa - \eta\tau_\pi}{2T} \right) \sigma_{\mu\nu} \left[\langle D\sigma^{\mu\nu} \rangle + \frac{1}{3} \sigma^{\mu\nu} (\nabla \cdot u) \right], \end{aligned} \quad (4.8)$$

which fixes the rate of entropy production slightly out of equilibrium for any conformal fluid up to one unknown parameter (A_1). Note that for any theory with $\kappa \neq 2\lambda_1$, the divergence of the non-equilibrium entropy current must differ from the simple expectation $\nabla_\mu S^\mu = \frac{\eta}{2T} \sigma_{\mu\nu} \sigma^{\mu\nu}$.

4.2 Non-Conformal Fluids

For non-conformal fluids, the non-equilibrium entropy current may be built out of all gradient structures. Keeping the form of the conformal entropy current intact, up to second order in gradients this leads to

$$\begin{aligned}
S_{\text{non-eq}}^\mu = & su^\mu + \frac{A_1}{4} \mathcal{S}_1 u^\mu + A_2 \mathcal{S}_2 u^\mu + A_3 \left(4\mathcal{S}_3 - \frac{1}{2} \mathcal{S}_1 + 2\mathcal{S}_2 \right) u^\mu + A_4 \mathcal{S}_4 u^\mu + A_5 \mathcal{S}_5 u^\mu \\
& + A_6 \mathcal{S}_6 u^\mu + B_1 \left(\frac{1}{2} \mathcal{V}_1^\mu + \frac{u^\mu}{4} \mathcal{S}_1 \right) + B_2 (\mathcal{V}_2^\mu - u^\mu \mathcal{S}_2) + B_3 \left(\mathcal{V}_3^\mu - \frac{1}{2} \mathcal{V}_1^\mu \right) + B_4 \mathcal{V}_4^\mu \\
& + B_5 \mathcal{V}_5^\mu + B_6 \mathcal{V}_6^\mu + \frac{A_7}{4} \left(\frac{2}{c_s^2} \mathcal{S}_7 u^\mu + \frac{4}{c_s^2} \mathcal{S}_3 u^\mu - \frac{2}{3c_s^2} \mathcal{S}_4 u^\mu + \frac{1}{c_s^2} \mathcal{S}_5 u^\mu + 2c_s^2 \mathcal{S}_6 u^\mu + 3\mathcal{V}_1^\mu \right. \\
& \left. + 6\mathcal{V}_2^\mu - 6\mathcal{V}_3^\mu + (4 - 6c_s^2) \mathcal{V}_4^\mu - 12c_s^2 \mathcal{V}_5^\mu - \frac{4}{3} \mathcal{V}_6^\mu \right), \quad (4.9)
\end{aligned}$$

where the cumbersome expression multiplying A_7 leads to a particularly simple divergence (see appendix B).

Since for non-conformal fluids c_s^2 may be a function of the entropy density, its derivative $c_s' \equiv \frac{d \ln c_s^2}{d \ln s}$ is in general non-vanishing. Similarly, the coefficient functions A_i, B_i now may contain logarithms or any power of the entropy density, so that in general $A_i' \equiv \frac{d \ln A_i}{d \ln s} \neq \frac{1}{3}$ (and the same for B_i). Using Eq. (3.4), the divergence of the non-equilibrium entropy current can be calculated, but the resulting expression (C.2) is suitably lengthy to be unenlightening except maybe for the expert reader, so it has been relegated to the appendix. Nevertheless, the principle of positivity of the entropy current divergence singles out conditions for the A_i, B_i as was the case for conformal fluids. One immediately finds

$$\begin{aligned}
A_5 = 0, \quad B_1 = 2A_3, \quad B_3 = 2(1 - 3c_s^2)A_3, \quad B_4 = -A_3(A_3' - 3c_s^2(1 - 2c_s^2)), \quad (4.10) \\
B_5 = B_2(B_2' - c_s^2) + (B_3' + c_s^2)B_3, \quad B_6 = 2A_6 - A_7(A_7' - c_s^2) + \frac{4}{3}(1 - 3c_s^2)A_3B_3'.
\end{aligned}$$

In addition, one can again go through the arguments as in section 4.1 to show that $\sigma_{\mu\nu} R^{\mu\nu}$ and $(\nabla \cdot u) R$ cannot form full squares because no other contribution to the divergence of the entropy current could offset their negative contribution to fourth order in derivatives. This leads to the conditions

$$A_3 = \frac{\kappa}{2T}, \quad \xi_5 = \frac{A_3 T}{3} (3A_3' - 1).$$

Finally, requiring that the cross-terms $u_\alpha u_\beta R^{\alpha<\mu\nu>\beta} \Omega^{<\mu\gamma} \Omega_{\nu}{}^\gamma$ and $u_\alpha u_\beta R^{\alpha\beta} \Omega_{\mu\nu} \Omega^{\mu\nu}$ cancel leads to

$$A_2 = -\frac{6\kappa(1-2c_s^2) - 2\kappa^* + \lambda_3}{4T}, \quad \frac{\xi_6 + \xi_3}{T} = -\frac{A_2}{3}(1-6c_s^2+3A_2') + 2(1-3c_s^2)A_3(2c_s^2-B_3').$$

Contrary to the situation for conformal fluids, fixing 8 of the 13 coefficient functions in the entropy current still does not seem to lead to a simple form for $\nabla_\mu S_{\text{non-eq}}^\mu$. It may be possible that a more detailed analysis of the effect of third order gradients S^μ , $\pi^{\mu\nu}$ and Π could result in further conditions on either the remaining five coefficient functions or some of the second order transport coefficients $\tau_\pi^*, \kappa^*, \lambda_4, \xi_1, \xi_2, \xi_3, \xi_4$.

5. Second Order Transport Coefficients: Known Results

The most general structure for the viscous energy-momentum tensor up to second order in gradients has been established in section 3. For conformal fluids at vanishing charge density, the energy-momentum tensor — and hence the equations of motion for viscous fluid dynamics — depend on 7 dimensionless numbers: the speed of sound and the transport coefficients. To first order in gradients, the only transport coefficient is the shear viscosity η , and to second order there are in general five additional coefficients: $\tau_\pi, \kappa, \lambda_1, \lambda_2, \lambda_3$. All of these transport coefficients seem to be independent, but so far $\lambda_3 = 0$ has been found for all examples of quantum field theories where it has been calculated, perhaps suggesting that there is another symmetry that has not been exploited yet.

For non-conformal fluids, the energy-momentum tensor can no longer be expressed in terms of dimensionless numbers because of the presence of an additional scale. Combinations involving transport coefficients that have vanishing mass dimension then are functions of, e.g., the entropy density. Also, there are 9 new independent transport coefficients in addition to the 6 for conformal fluids: to first order in gradients the bulk viscosity ζ and to second order $\tau_\Pi, \tau_\pi^*, \kappa^*, \lambda_4, \xi_1, \xi_2, \xi_3, \xi_4$, while ξ_5, ξ_6 are specified in terms of the other transport coefficients and their derivatives.

Many studies have been published on the value of the first order transport coefficients η, ζ , and hence I will focus here exclusively on second order transport coefficients and review the existing results for relativistic field theories. Of particular interest are the values of the “relaxation times” τ_π, τ_Π in the shear and bulk channel, respectively, since these have implications for numerical solutions of relativistic viscous fluid dynamics.

For conformal fluids in 3+1 spacetime dimensions, τ_π is known for quantum field theories at weak coupling $\lambda \ll 1$ [23], and for a particular quantum field example

($\mathcal{N} = 4$ SYM theory) at very large coupling $\lambda \gg 1$ [24], and for infinite coupling $\lambda \rightarrow \infty$ [14, 25]:

$$\lim_{\lambda \rightarrow 0} \tau_\pi \sim 5.9 \frac{\eta}{\varepsilon + P}, \quad \lim_{\lambda \rightarrow \infty} \tau_\pi \sim \left(4 - 2 \ln 2 + \frac{375}{8} \zeta(3) \lambda^{-3/2}\right) \frac{\eta}{\varepsilon + P}.$$

For non-conformal fluids, τ_Π is known only in one particular example of a strongly coupled field theory [26] where

$$\tau_\Pi = \tau_\pi, \quad \zeta = 2\eta \left(\frac{1}{3} - c_s^2 \right), \quad \tau_\pi = (4 - 2 \ln 2) \frac{\eta}{\varepsilon + P}.$$

Remarkably, for these cases, the maximal propagation speeds (3.6) turn out to be less than unity, indicating that the second order fluid dynamic equations of motion obey causality [19]. Furthermore, it was shown recently that causality in second order fluid dynamics is guaranteed by the causality of the underlying quantum field theory for the conformal field theory example of $\mathcal{N} = 2$ SYM at strong coupling [27]. This seems to suggest that second order fluid dynamics may be a particularly useful approximation of quantum field theories in the appropriate limit.

The parameter λ_1 is known for quantum field theories at weak coupling [23], and for $\mathcal{N} = 4$ SYM theory at very large, and infinite coupling [28, 14, 25]:

$$\lim_{\lambda \rightarrow 0} \lambda_1 \sim 5.2 \frac{\eta^2}{\varepsilon + P}, \quad \lim_{\lambda \rightarrow \infty} \lambda_1 \sim \left(2 + \frac{215}{4} \zeta(3) \lambda^{-3/2}\right) \frac{\eta^2}{\varepsilon + P}.$$

The parameter λ_2 is known for quantum field theories at weak coupling [23], and for $\mathcal{N} = 4$ SYM theory at infinite coupling [25]:

$$\lim_{\lambda \rightarrow 0} \lambda_2 \sim -2\eta\tau_\pi, \quad \lim_{\lambda \rightarrow \infty} \lambda_2 \sim -\ln 2 \frac{\eta}{\pi T}.$$

In attempts to calculate λ_3 , the parameter has been found to vanish at weak and strong coupling [23, 25].

The parameter κ has been calculated in weakly coupled $SU(N)$ gauge theory [29] and for $\mathcal{N} = 4$ SYM theory for large, and for infinite coupling [14, 24]:

$$\lim_{\lambda \rightarrow 0} \kappa \sim \frac{5s}{8\pi^2 T}, \quad \lim_{\lambda \rightarrow \infty} \kappa \sim \frac{s}{4\pi^2 T} \left(1 - \frac{145}{8} \zeta(3) \lambda^{-3/2}\right).$$

Note that $\kappa \neq 2\lambda_1$, except for $\mathcal{N} = 4$ SYM theory at infinite coupling.

For the strongly coupled theory example of Ref. [26], all second order transport coefficients for a particular class of non-conformal fluids may be extracted:

$$\begin{aligned} \kappa^* &= -\frac{\kappa}{2c_s^2} (1 - 3c_s^2), \quad \tau_\pi^* = -\tau_\pi (1 - 3c_s^2), \quad \lambda_4 = 0, \\ \xi_1 &= \frac{\lambda_1}{3} (1 - 3c_s^2), \quad \xi_2 = \frac{2\eta\tau_\pi c_s^2}{3} (1 - 3c_s^2), \quad \xi_3 = \frac{\lambda_3}{3} (1 - 3c_s^2), \quad \xi_4 = 0, \\ \xi_5 &= \frac{\kappa}{3} (1 - 3c_s^2), \quad \xi_6 = \frac{\kappa}{3c_s^2} (1 - 3c_s^2). \end{aligned} \quad (5.1)$$

As a non-trivial consistency check, one can proceed to evaluate the relations between ξ_5, ξ_6 and the other transport coefficient derived in section 4.2. Using $T = s^{c_s^2}$ and $\kappa \propto s/T$, one finds

$$\xi_5 = \frac{A_3 T}{3} (3A'_3 - 1) = \frac{\kappa}{3} (1 - 3c_s^2),$$

$$\xi_6 + \xi_3 = -\frac{T}{3} A_2 (1 - 6c_s^2 + 3A'_2) + 2T(1 - 3c_s^2) A_3 (2c_s^2 - B'_3) = \frac{1 - 3c_s^2}{3c_s^2} (\kappa + \lambda_3 c_s^2),$$

which precisely matches the values from Ref. [26]. It would be interesting to check these relations for other examples of non-conformal field theories (e.g., by compactification of conformal field theories suggested in [30]).

Furthermore, one can use the above results for the conformal fluid transport coefficients for infinitely strongly coupled $\mathcal{N} = 4$ SYM theory and evaluate the three relations (4.3), (4.4), (4.7):

$$\frac{(\pi T)^2}{4\pi\eta} A_2 = -\frac{1}{8}, \quad \frac{(\pi T)^2}{4\pi\eta} A_3 = \frac{1}{8}, \quad \frac{(\pi T)^2}{4\pi\eta} B_1 = \frac{1}{4}. \quad (5.2)$$

These values precisely correspond to those for the entropy current derived from the horizon of a black hole in AdS_5 (see Ref. [22]). This serves as another consistency-check of the above approach.

It would be interesting to calculate the entropy current from the black hole horizon corresponding to the geometry considered in Ref. [26] and check that its form fulfills the conditions on $B_1, B_3, B_4, B_5, B_6, A_2, A_3, A_5$ found in section 4.2.

Also, it could be possible to constrain the coefficient A_1 for $\mathcal{N} = 4$ SYM using existing results in a gradient expansion beyond second order [31].

6. Discussion and Conclusions

In this work, I have derived the structure of the energy-momentum tensor and non-equilibrium entropy current for a relativistic viscous fluid in curved space at vanishing charge density up to second order in gradients. I found that in this case there are 15 possible transport coefficients multiplying second order gradient terms (10 of which vanish for conformal fluids), in addition to the bulk and shear viscosity coefficients arising at first order. Requiring the divergence of the (non-equilibrium) entropy current to be positive definite, two of these 15 coefficients are found not to be independent, and their relation to the other coefficients is specified.

Also, the most general form of the entropy current out of equilibrium allows 13 possible coefficient functions (8 of which vanish for conformal fluids) multiplying terms

of second order in gradients. Requiring positivity fixes 8 (conformal fluids: three) of these coefficient functions.

These results resolve any ambiguities in the structure of the viscous fluid dynamic energy-momentum tensor and thereby clear the path for an extraction of the (second order) transport coefficients from quantum field theories. For instance, using Eq. (3.2) and Eq. (3.3) one can derive Kubo-like formulas for correlators in the fluid dynamics regime for non-conformal fluids. In addition, knowledge of the correlators and spectral functions to second order in gradients may be vital to extract first order gauge theory transport coefficients from lattice Quantum-Chromodynamics [32].

Also, knowing the structure of the viscous fluid dynamic equations may be required to extend conformal fluid dynamics simulations of heavy-ion collisions [33, 34, 35, 36] to reliably include effects from bulk viscosity. It could furthermore help to clarify the role of non-linear viscous damping of the r-mode instability in rotating neutron stars [37, 38, 39]

Moreover, the agreement (5.2) of the non-equilibrium entropy current from fluid dynamics and that derived from the horizon area of a black hole support the hope of linking these two concepts in more detail, which might have wide ranging consequences. In this context it is curious to note that for conformal fluids, where one parameter (A_1) could not be fixed by requiring positivity of the entropy current alone, a diffeomorphism ambiguity prevents fixing of the same parameter on the gravity side. It is unknown whether A_1 (and hence the fluid entropy current) cannot be fixed in principle in fluid dynamics, or whether this is only a manifestation of our lack of ingenuity (cf. the discussion in Ref. [40] for the gravitational standpoint). However, I believe A_1 should have a definite value, since nature must know the amount of entropy production when taking a system out of equilibrium.

Furthermore, it seems that genuine non-equilibrium contributions to the entropy current are necessary to ensure consistency of viscous fluid dynamics with the requirement that entropy may never decrease. However, it is unknown to me whether the presence of these non-equilibrium contributions to the entropy current is an experimentally established fact. If not, relations such as Eq. (4.8) could potentially be used to attempt such an experimental verification, since it was found that the divergence of the non-equilibrium entropy current differs from the “naive” expectation $\nabla_\mu S^\mu = \frac{\eta}{2T} \sigma_{\mu\nu} \sigma^{\mu\nu}$. E.g., this could be done by measuring the entropy of a conformal fluid in equilibrium, then subjecting the fluid to a perturbation $\sigma_{\mu\nu}$ (ideally one that is tuned such that $\langle D\sigma^{\mu\nu} \rangle \simeq -\frac{1}{3} \sigma^{\mu\nu} (\nabla \cdot u)$ to eliminate the dependence on the unknown coefficient A_1), switching off the perturbation and letting the system relax back to equilibrium, then measuring again its entropy. Setups reminiscent of this proposal are being pursued in cold atom experiments [41], but it is unclear to me whether these could be adapted to

test for non-equilibrium entropy effects.

Finally, it should be possible to extend my analysis to include the case of non-vanishing charge density, allowing for the presence of charge/heat diffusion by accounting for gradients in the chemical potential, as well as including structures built out of the Levi-Civita symbol that break parity. This would be the last step in determining the equations of motion for a general one-component relativistic viscous fluid.

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A. Conformal Transformations

Here I collect the behavior of terms up to second order in gradients under Weyl rescalings (2.7). For the Christoffel symbols one finds

$$\Gamma_{\mu\nu}^{\lambda} \rightarrow \Gamma_{\mu\nu}^{\lambda} - (\delta_{\nu}^{\lambda} \partial_{\mu} w + \delta_{\mu}^{\lambda} \partial_{\nu} w - g_{\mu\nu} \partial^{\lambda} w) ,$$

from which the behavior of the covariant derivative of the fluid velocity can be calculated to be

$$\nabla_{\mu} u_{\nu} \rightarrow e^{-w} (\nabla_{\mu} u_{\nu} + u_{\mu} \partial_{\nu} w - g_{\mu\nu} D w) .$$

This in turn implies

$$\begin{aligned} \sigma_{\mu\nu} &\rightarrow e^{-w} \sigma_{\mu\nu} , \\ \Omega_{\mu\nu} &\rightarrow e^{-w} \Omega_{\mu\nu} , \\ \nabla \cdot u &\rightarrow e^w (\nabla \cdot u - 3 D w) , \end{aligned}$$

which can be used to derive

$$\begin{aligned} \nabla_{\nu}^{\perp} \sigma^{\mu\nu} &\rightarrow \nabla_{\nu}^{\perp} \sigma^{\mu\nu} - 2 \sigma^{\mu\nu} \nabla_{\nu}^{\perp} w , \\ \nabla_{\nu}^{\perp} \Omega^{\mu\nu} &\rightarrow \nabla_{\nu}^{\perp} \Omega^{\mu\nu} . \end{aligned}$$

Furthermore, using the Christoffel symbols one can calculate the transformation of the Riemann tensor to be

$$R_{\mu\sigma\nu}^{\lambda} \rightarrow R_{\mu\sigma\nu}^{\lambda} - \delta_{\nu}^{\lambda} \nabla_{\mu} \partial_{\sigma} w + \delta_{\sigma}^{\lambda} \nabla_{\mu} \partial_{\nu} w + g_{\mu\nu} \nabla^{\lambda} \partial_{\sigma} w - g_{\mu\sigma} \nabla^{\lambda} \partial_{\nu} w ,$$

so that

$$\begin{aligned}
u_\lambda u^\nu R^\lambda_{<\mu\sigma>\nu} &\rightarrow u_\lambda u^\nu R^\lambda_{<\mu\sigma>\nu} + \nabla_{<\mu} \partial_{\sigma>} w, \\
R_{\mu\nu} &\rightarrow R_{\mu\nu} + 2\nabla_\mu \partial_\nu w + g_{\mu\nu} \nabla^\lambda \partial_\lambda w, \\
R &\rightarrow e^{2w} (R + 6\nabla^\lambda \partial_\lambda w).
\end{aligned} \tag{A.1}$$

Finally, one has

$$\nabla_\mu^\perp \ln s \rightarrow \nabla_\mu^\perp \ln s + 3\nabla_\mu^\perp w,$$

and the above result for the Christoffel symbols implies

$$\begin{aligned}
\nabla_{<\mu}^\perp \nabla_{>\nu}^\perp \ln s &\rightarrow \nabla_{<\mu}^\perp \nabla_{>\nu}^\perp \ln s + 3\nabla_{<\mu}^\perp \partial_{>\nu} w + 2\nabla_{<\mu}^\perp \ln s \nabla_{>\nu}^\perp w + \frac{3}{2} \sigma_{\mu\nu} Dw, \\
\nabla_\mu^\perp \nabla_\perp^\mu \ln s &\rightarrow e^{2w} (\nabla_\mu^\perp \nabla_\perp^\mu \ln s + 3\nabla_\mu^\perp \partial^\mu w + 3(\nabla \cdot u) Dw - \nabla_\mu^\perp \ln s \nabla_\perp^\mu w).
\end{aligned}$$

B. Useful Identities

This appendix contains some useful identities that were, e.g., used in calculating the divergence of the non-equilibrium entropy current.

Time derivatives of various scalars and vectors:

$$\begin{aligned}
\Omega_{\mu\nu} D\Omega^{\mu\nu} &= -\frac{2-3c_s^2}{3} \Omega_{\mu\nu} \Omega^{\mu\nu} (\nabla \cdot u) - \Omega_{\mu\nu} \Omega^{\beta\nu} \sigma^\mu_\beta, \\
D\nabla_\mu^\perp \ln s &= -\nabla_\mu^\perp (\nabla \cdot u) - \left(\frac{1}{2} \sigma_{\mu\beta} + \Omega_{\mu\beta} \right) \nabla_\perp^\beta \ln s + \left(c_s^2 - \frac{1}{3} \right) (\nabla \cdot u) \nabla_\mu^\perp \ln s \\
&\quad - u_\mu c_s^2 \nabla_\alpha^\perp \ln s \nabla_\perp^\alpha \ln s, \\
D\nabla_\mu^\perp \nabla_\perp^\mu \ln s &= -\nabla_\mu^\perp \nabla_\perp^\mu (\nabla \cdot u) - u^\nu R_{\mu\nu} \nabla_\perp^\mu \ln s \\
&\quad - \nabla_\perp^\mu \nabla_\perp^\nu \ln s \left(\sigma_{\mu\nu} + \Delta_{\mu\nu} (\nabla \cdot u) \frac{2-3c_s^2}{3} \right) \\
&\quad + c_s^2 \nabla_\perp^\mu \ln s \nabla_\perp^\nu \ln s \left(\sigma_{\mu\nu} - \Delta_{\mu\nu} (\nabla \cdot u) \frac{1+3c_s^2-3(d \ln c_s^2/d \ln s)}{3} \right) \\
&\quad + \nabla_\perp^\mu \ln s \left(\frac{6c_s^2-1}{3} \nabla_\mu^\perp (\nabla \cdot u) - \frac{1}{2} \nabla_\perp^\beta \sigma_{\beta\mu} - \nabla_\perp^\beta \Omega_{\beta\mu} \right), \\
D[R + 2u^\alpha u^\beta R_{\alpha\beta}] &= 2 \left[\nabla_\mu^\perp (u_\nu R^{\mu\nu}) - R^{\mu\nu} \left(\frac{1}{2} \sigma_{\mu\nu} + \frac{1}{3} \Delta_{\mu\nu} (\nabla \cdot u) \right) - 2c_s^2 u_\nu R^{\mu\nu} \nabla_\mu^\perp \ln s \right].
\end{aligned} \tag{B.1}$$

Identities involving curvature tensors and vorticity:

$$\begin{aligned}
R^\lambda_{\mu\sigma\nu} u_\lambda &= (\nabla_\nu \nabla_\sigma - \nabla_\sigma \nabla_\nu) u_\mu, \\
R_{\lambda\sigma} u^\lambda &= 2c_s^2 \Omega_{\mu\sigma} \nabla_\perp^\mu \ln s + \nabla_\perp^\mu (\nabla_\sigma^\perp u_\mu) - D_\sigma (\nabla \cdot u)
\end{aligned}$$

$$\begin{aligned}
& +c_s^2 u_\sigma \left[\nabla_\mu^\perp \nabla_\perp^\mu \ln s - \nabla_\mu^\perp \ln s \nabla_\perp^\mu \ln s \left(c_s^2 - \frac{d \ln c_s^2}{d \ln s} \right) \right], \\
& \nabla_\perp^\alpha u^\lambda R_{\alpha\lambda} = 2c_s^2 \nabla_\alpha^\perp \Omega^{\mu\alpha} \nabla_\mu^\perp \ln s + \frac{1}{2} \nabla_\alpha^\perp \nabla_\mu^\perp \sigma^{\mu\alpha} - \frac{2}{3} \nabla_\mu^\perp \nabla_\perp^\mu (\nabla \cdot u) - 2\Omega_{\mu\nu} \Omega^{\beta\nu} \sigma_\beta^\mu \\
& + (\nabla \cdot u) \left[\Omega_{\mu\nu} \Omega^{\mu\nu} \left(3c_s^2 - \frac{1}{3} \right) - \frac{1}{4} \sigma_{\mu\nu} \sigma^{\mu\nu} - u_\mu u_\nu R^{\mu\nu} \right], \\
& \Omega_{\mu\nu} \nabla_\perp^\mu \nabla_\perp^\nu \ln s = -\Omega_{\mu\nu} \Omega^{\mu\nu} (\nabla \cdot u), \\
& \nabla_\beta^\perp \nabla_\alpha^\perp \Omega^{\alpha\beta} = \frac{4 - 3c_s^2}{3} \Omega_{\mu\nu} \Omega^{\mu\nu} (\nabla \cdot u) + 2\Omega_{\mu\nu} \Omega^{\beta\nu} \sigma_\beta^\mu.
\end{aligned}$$

Identities for scalar \mathcal{S}_7 :

$$\begin{aligned}
& \frac{1}{4} \left(\frac{2}{c_s^2} \mathcal{S}_7 u^\mu + \frac{4}{c_s^2} \mathcal{S}_3 u^\mu - \frac{2}{3c_s^2} \mathcal{S}_4 u^\mu + \frac{1}{c_s^2} \mathcal{S}_5 u^\mu + 2c_s^2 \mathcal{S}_6 u^\mu \right. \\
& \left. + 3\mathcal{V}_1^\mu + 6\mathcal{V}_2^\mu - 6\mathcal{V}_3^\mu + (4 - 6c_s^2) \mathcal{V}_4^\mu - 12c_s^2 \mathcal{V}_5^\mu - \frac{4}{3} \mathcal{V}_6^\mu \right) = \\
& u^\mu \left(c_s^2 \nabla_\alpha^\perp \ln s \nabla_\perp^\alpha \ln s + \nabla_\alpha^\perp \nabla_\perp^\alpha \ln s \right) + \nabla_\perp^\mu (\nabla \cdot u) + \sigma^{\mu\alpha} \nabla_\alpha^\perp \ln s - \frac{1}{3} (\nabla \cdot u) \nabla_\perp^\mu \ln s.
\end{aligned} \tag{B.2}$$

C. $\nabla_\mu S_{\text{non-eq}}^\mu$ for Non-Conformal Fluids

The explicit divergence of Eq. (4.9) is

$$\begin{aligned}
& \nabla_\mu S_{\text{non-eq}}^\mu = \\
& \nabla_\mu^\perp \nabla_\nu^\perp \sigma^{\mu\nu} \frac{(-2A_3 + B_1)}{2} + \nabla_\mu^\perp \sigma^{\mu\nu} \nabla_\nu^\perp \ln s \left(-2c_s^2 A_3 + \frac{B_1}{2} (c_s^2 + B_1') - c_s^2 B_3 + B_4 \right) \\
& + \frac{2}{3} \nabla_\mu^\perp \nabla_\perp^\mu (\nabla \cdot u) (2(1 - 3c_s^2) A_3 - B_3) + A_5 D R \\
& + \nabla_\alpha^\perp \Omega^{\mu\alpha} \nabla_\mu^\perp \ln s (B_2(B_2' - c_s^2) + B_3(B_3' + c_s^2) - B_5) \\
& + \frac{1}{3} \nabla_\perp^\mu (\nabla \cdot u) \nabla_\mu^\perp \ln s (-4c_s^2(1 - 3c_s^2) A_3 - 6A_6 + 3A_7(A_7' - c_s^2) - 2B_3(B_3' - c_s^2) + 3B_6) \\
& + \sigma_{\mu\nu} \left[\frac{\eta}{2T} \sigma^{\mu\nu} + R^{\mu\nu} \left(-\frac{\kappa}{2T} + A_3 \right) + u_\alpha u_\beta R^{\alpha<\mu\nu>\beta} \left(\frac{\kappa - \kappa^* - \eta\tau_\pi}{T} + A_1 + B_1 - 2A_3 \right) \right] \\
& + \Omega^\mu{}_\alpha \Omega^{\nu\alpha} \left(-\frac{\lambda_3 + 2\eta\tau_\pi}{2T} + A_1 - 2A_2 - 2A_3 + B_1 - 2B_3 \right) \\
& + c_s^2 \nabla_\perp^{<\mu} \nabla_\perp^{\nu>} \ln s \left(\frac{\eta\tau_\pi}{T} - A_1 - 2A_3 - B_3 + c_s^{-2} B_4 \right) \\
& - \frac{1}{4} \sigma^\mu{}_\lambda \sigma^{\nu\lambda} \left(\frac{2\lambda_1 - \eta\tau_\pi}{T} + A_1 + B_1 - 2A_3 \right) + \sigma^{\mu\nu} \frac{(\nabla \cdot u)}{12} \left(\frac{2\eta(\tau_\pi - \tau_\pi^*) - 12\xi_1 + 3\zeta\tau_\Pi}{T} \right. \\
& \quad \left. - A_1(1 + 3A_1') + A_3(4 + 6A_3') - 6A_4 - B_1(4 + 3B_1') \right) \\
& + \nabla_\perp^{<\mu} \ln s \nabla_\perp^{\nu>} \ln s \left(-\frac{2c_s^2(c_s^2 - c_s')\eta\tau_\pi + \lambda_4}{2T} + c_s^2(c_s^2 - c_s') A_1 + 2c_s^2 c_s' A_3 - A_6 \right)
\end{aligned}$$

$$\begin{aligned}
& +A_7(A'_7 - c_s^2) + c_s^2 B'_1 B_1 + c_s^2(c_s^2 - B'_3 - c'_s)B_3 + (B'_4 - c_s^2)B_4] \cdot \\
& + (\nabla \cdot u) \left[\frac{\zeta}{T} (\nabla \cdot u) + R \left(-\frac{\xi_5}{T} + (1 - A'_5)A_5 + \frac{1}{3}(3A'_3 - 1)A_3 \right) \right. \\
& + u_\alpha u_\beta R^{\alpha\beta} \left(-\frac{\xi_6 - \zeta\tau_\Pi}{T} - \frac{2}{3}(1 - 3A'_3)A_3 - 2A_4 \right) \\
& + (\nabla \cdot u)^2 \left(\frac{\zeta\tau_\Pi - 3\xi_2}{3T} + \frac{2}{9}(1 - 3A'_3)A_3 + \frac{1}{3}(1 - 3A'_4)A_4 \right) \\
& + \Omega_{\alpha\beta}\Omega^{\alpha\beta} \left(-\frac{\xi_3 + \zeta\tau_\Pi}{T} - \frac{1}{3}(1 - 6c_s^2 + 3A'_2)A_2 + \frac{2}{3}(2 - 3c_s^2 - 3A'_3)A_3 + 2A_4 \right. \\
& \quad \left. - (c_s^2 - B'_2)B_2 - \frac{1}{3}(1 - 9c_s^2)B_3 - B_5 \right) \\
& + \nabla_\mu^\perp \nabla_\perp^\mu \ln s \, c_s^2 \left(\frac{\zeta\tau_\Pi}{T} + 4(c_s^2 - c'_s - A'_3)A_3 - 2A_4 + (1 - c_s^{-2}A'_7)A_7 + c_s^{-2}B_6 \right) \\
& + \nabla_\mu^\perp \ln s \nabla_\perp^\mu \ln s \left(-\frac{c_s^2(c_s^2 - c'_s)\zeta\tau_\Pi + \xi_4}{T} + \frac{2}{3}c_s^2(c_s^2 - 2c'_s - 3c_s^2A'_3)A_3 \right. \\
& \quad \left. + 2c_s^2(c_s^2 - c'_s)A_4 + \frac{1}{3}(1 + 6c_s^2 - 3A'_6)A_6 + \frac{1}{3}(1 + 3c_s^2)(c_s^2 - A'_7)A_7 - (c_s^2 - B'_6)B_6 \right) \Big] , \tag{C.1}
\end{aligned}$$

where it is recalled that $c'_s = \frac{d \ln c_s^2}{d \ln s}$, $A'_i = \frac{d \ln A_i}{d \ln s}$, and $B'_i = \frac{d \ln B_i}{d \ln s}$.

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